Summary of Lecture 5

- In lecture 5 we learnt how to pick the reproduction levels for the given thresholds.
- We learnt how to design MSQE optimal (Lloyd-Max) quantizers.
- We reviewed linear systems, linear shift invariant systems and the convolution sum.

LSI Systems and Convolution

$\mathcal{S}$ is a linear shift invariant system with input-output relationship $\mathcal{H}$.

$$\begin{aligned}\mathcal{H}(A(i,j)) &= \mathcal{H}(\sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} A(k,l)\delta(i-k,j-l)) \\ &= \sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} A(k,l)\mathcal{H}(\delta(i-k,j-l))\end{aligned}$$

$h(i,j) = \mathcal{H}(\delta(i,j))$ the impulse response of the system $\mathcal{S}$.

$$\begin{aligned}\mathcal{H}(\delta(i-k,j-l)) &= h(i-k,j-l) \\ \mathcal{H}(A(i,j)) &= \sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} A(k,l)h(i-k,j-l)\end{aligned}$$

which is the the convolution sum.

- Everything about the LSI system $\mathcal{S}$ is “in” $h(i,j)$.

Convolution

$\mathbf{B} = \mathbf{A} \otimes \mathbf{h}$:

$$B(i,j) = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} A(k,l) h(i-k, j-l) \quad (1)$$

Properties:

- $\mathbf{A} \otimes \mathbf{h} = \mathbf{h} \otimes \mathbf{A}$.
- $\mathbf{A} \otimes \delta = \mathbf{A}$.
- Finite extent $2-D$ sequences $\mathbf{A}$ $(N_1 \times M_1)$, $\mathbf{h}$ $(N_2 \times M_2)$:
  (for e.g., $A(i,j) \neq 0$, $0 \leq i \leq N_1 - 1$, $0 \leq j \leq M_1 - 1$, etc.)
  - $\mathbf{C} = \mathbf{A} \otimes \mathbf{h}$ is $(N_1 + N_2 - 1) \times (M_1 + M_2 - 1)$.

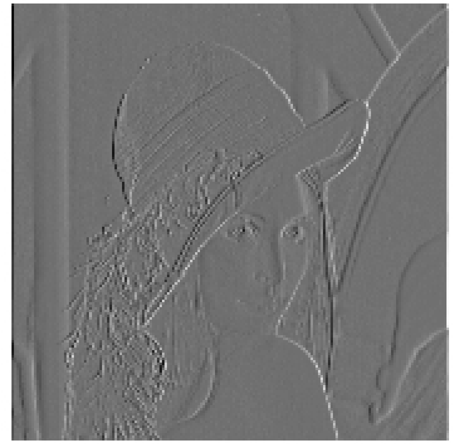
Convolution and Linear Filtering

- $\mathbf{B} = \mathbf{A} \otimes \mathbf{h}$
  - "$\mathbf{A}$ is convolved with $\mathbf{h}$ to produce $\mathbf{B}$".
  - "$\mathbf{A}$ is linearly filtered with $\mathbf{h}$ to produce $\mathbf{B}$".
- Of course using $\mathbf{A} \otimes \mathbf{h} = \mathbf{h} \otimes \mathbf{A}$ we can also say:
  - "$\mathbf{h}$ is convolved with $\mathbf{A}$ to produce $\mathbf{B}$".
  - "$\mathbf{h}$ is linearly filtered with $\mathbf{A}$ to produce $\mathbf{B}$".
- We can solve many interesting image processing problems by cleverly choosing the "filter" $\mathbf{h}$ and filtering the image $\mathbf{A}$.

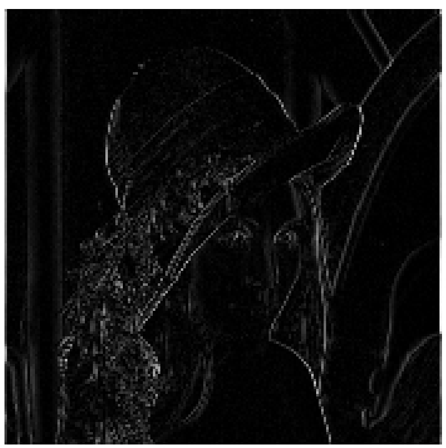
Example

A

B=A $\otimes$ h (normalized)



C=abs(A $\otimes$ h) (normalized)

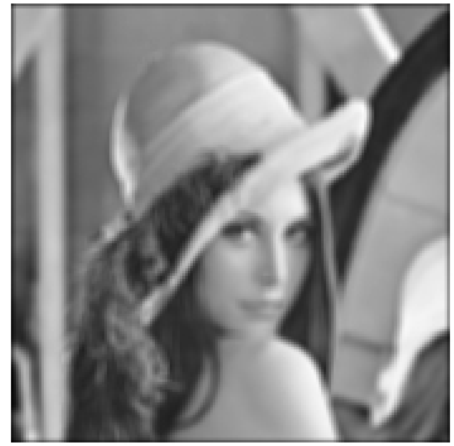


image((C>T)*255) (T=25)

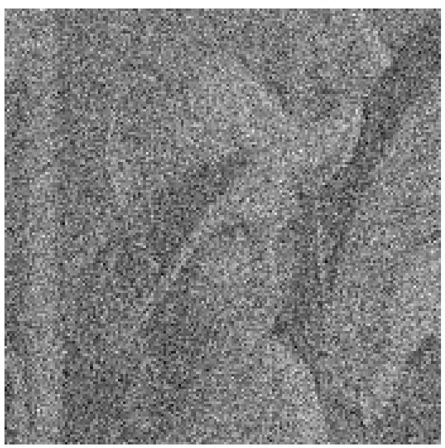
Example - contd.

A

$A \otimes g$ (normalized)



B=A+100*randn(512,512) (normalized)



$B \otimes g$ (normalized)

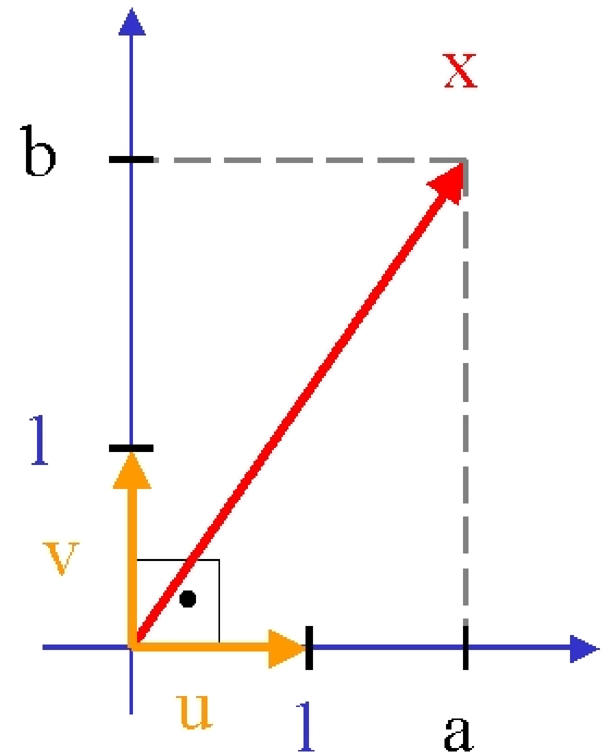
The Fourier Transform of 2-D Sequences

We will now review the Fourier Transform of 2-D sequences.
Motivation:

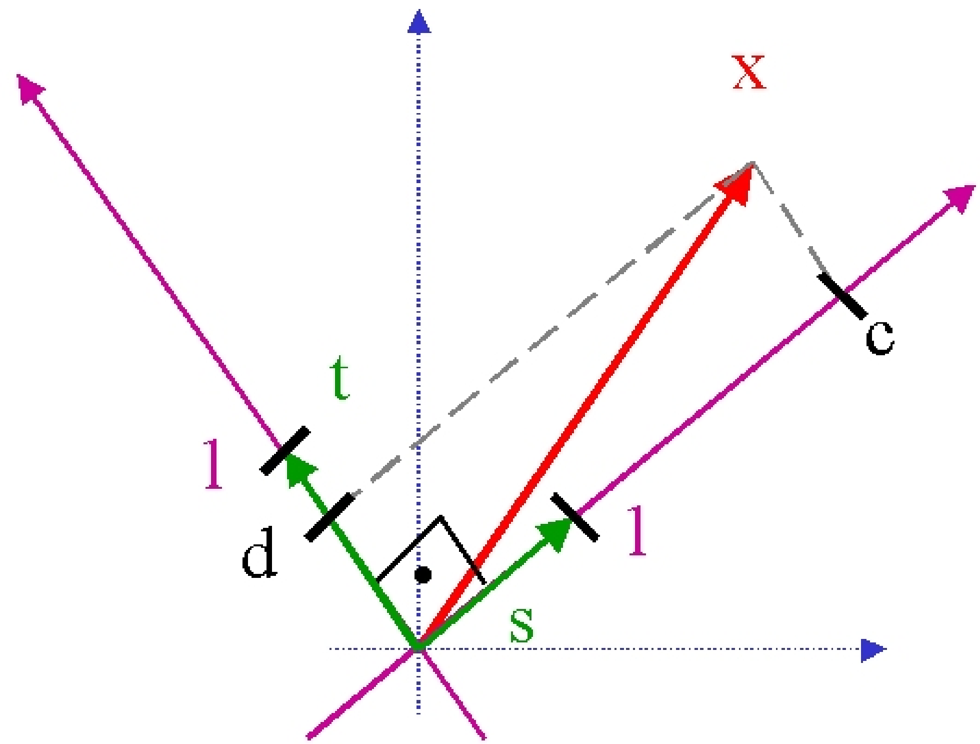
- The convolution operation takes on a very special form in 2-D Fourier transform "domain".
- The 2-D Fourier transform of images will reveal interesting properties that are shared by many images.
  - This will allow us to distinguish natural images from "non-images" (such as noise).
  - We will be able to say what "kind" of linear filter is good for a certain processing application.
- The effect of sampling operations are understood more clearly in 2-D Fourier transform "domain".
- This class will mostly discuss the required "tools".

Intuition - Orthogonal Coordinate Systems

Original Coordinate System



Rotated Coordinate System



u , v , s , t : unit vectors

$x = a\,u + b\,v = c\,s + d\,t$

Definition

The 2-D Fourier Transform of a 2-D sequence $\mathbf{A}$, $\mathcal{F}(\mathbf{A})$ is defined as:

$$\begin{aligned} \mathcal{F}(\mathbf{A}) &= F_A(w_1, w_2) \\ &= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} A(m,n) e^{-j(mw_1 + nw_2)} \qquad -\pi \leq w_1, w_2 < \pi \end{aligned} \tag{2}$$

$\mathbf{A}$ can be *recovered* back from its transform $F_A(w_1, w_2)$
via the inverse 2-D Fourier Transform $\mathcal{F}^{-1}(\mathbf{A})$:

$$\begin{aligned} A(m,n) &= \mathcal{F}^{-1}(\mathbf{A}) \\ &= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F_A(w_1, w_2) e^{+j(mw_1 + nw_2)} dw_1 dw_2 \end{aligned} \tag{3}$$

- w_1, w_2 vary in a continuum, i.e., the interval $[-\pi, \pi)$.
- $e^{j(mw_1 + nw_2)} = cos(mw_1 + nw_2) + jsin(mw_1 + nw_2)$.
- $\mathbf{A} \stackrel{\mathcal{F}}{\leftrightarrow} F_A$

Intuition - contd.

Continuing with the previous intuition, consider the impulse representation of 2-D sequences:

$$A(k,l) = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} A(m,n)\delta(m-k,n-l)$$

and their Fourier transforms:

$$F_A(w_1,w_2) = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} A(m,n)e^{-j(mw_1+nw_2)}$$

These are actually the representations of the same sequence $\mathbf{A}$ in two orthogonal coordinate systems:

- The first coordinate system has basis "vectors" given by the $\delta(m-k,n-l)$.
- The second coordinate system has basis "vectors" given by the $e^{-j(mw_1+nw_2)}$.
- The sums are inner or "scalar" products.

Real-Complex Parts and Symmetry

- In general $F_A(w_1, w_2)$ is complex valued.
- Since we will be mainly be considering real 2-d sequences we can note some symmetry properties by using the inverse Fourier transform relationship.

$$A(m,n) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_A(w_1,w_2)e^{+j(mw_1+nw_2)}dw_1 dw_2$$

If $\mathbf{A}$ is real then:

$$F_A(w_1,w_2) = F_A^*(-w_1,-w_2) \quad (4)$$
$$|F_A(w_1,w_2)| = |F_A(-w_1,-w_2)| \quad (5)$$
$$\angle F_A(w_1,w_2) = -\angle F_A(-w_1,-w_2) \quad (6)$$
$$\Re(F_A(w_1,w_2)) = \Re(F_A(-w_1,-w_2)) \quad (7)$$
$$\Im(F_A(w_1,w_2)) = -\Im(F_A(-w_1,-w_2)) \quad (8)$$

Periodicity

$$F_A(w_1, w_2) = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} A(m,n) e^{-j(mw_1+nw_2)} \qquad -\pi \leq w_1, w_2 < \pi$$

- $F_A(w_1, w_2)$ is periodic in w_1, w_2 with period 2π, i.e., for all integers k, l:

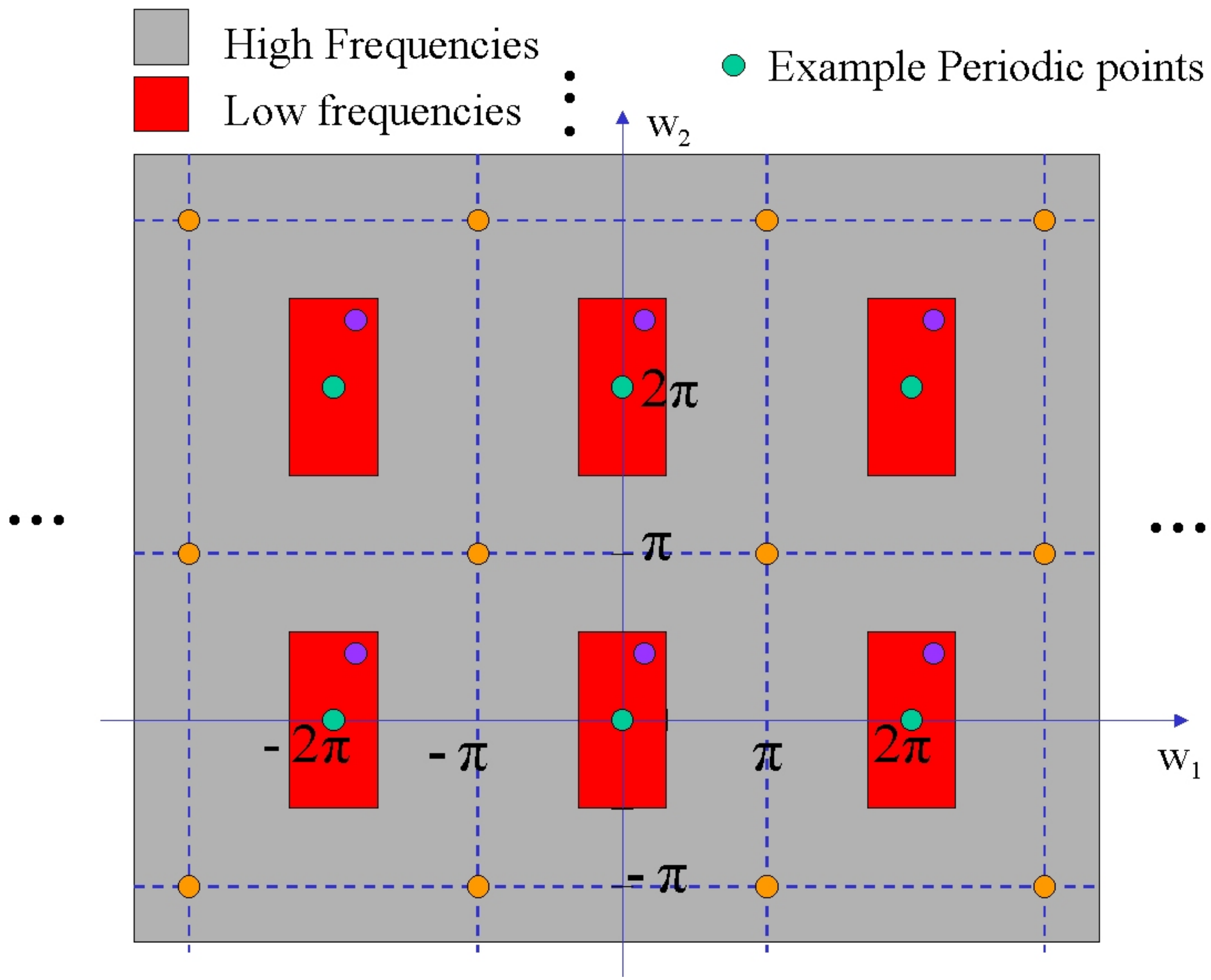
$$F_A(w_1 + k2\pi, w_2 + l2\pi) = F_A(w_1, w_2) \tag{9}$$

  To see this consider:

$$\begin{aligned} e^{-j(m(w_1+k2\pi)+n(w_2+l2\pi))} &= e^{-j(mw_1+nw_2)} e^{-jk2\pi} e^{-jl2\pi} \\ &= e^{-j(mw_1+nw_2)} \quad \forall \quad \text{integers} \quad k, l \end{aligned}$$

- $e^{j(mw_1+nw_2)} = cos(mw_1 + nw_2) + jsin(mw_1 + nw_2)$.
  w_1, w_2 the frequencies of the periodic trigonometric functions.

Example


High Frequencies
Low frequencies
Example Periodic points
w2
2π
π
- 2π
- π
π
2π
w1
- π

Shifting and Modulation

- Shifting:

$$\begin{aligned}\mathcal{F}(A(m-m_0,n-n_0)) &= \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} A(m-m_0,n-n_0)e^{-j(mw_1+nw_2)} \\ &= \sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} A(k,l)e^{-j((k+m_0)w_1+(l+n_0)w_2)} \\ &= e^{-j(m_0w_1+n_0w_2)}\sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} A(k,l)e^{-j(kw_1+lw_2)} \qquad (10)\end{aligned}$$

$A(m-m_0,n-n_0) \overset{\mathcal{F}}{\leftrightarrow} e^{-j(m_0w_1+n_0w_2)}F_A(w_1,w_2)$.

- Similarly, modulation:

$$e^{j(mw_{01}+mw_{02})}A(m,n) \overset{\mathcal{F}}{\leftrightarrow} F_A(w_1-w_{01},w_2-w_{02}) \qquad (11)$$

Inner Product and Energy Conservation

- Conservation of the inner product:

$$
\begin{aligned}
&\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} A(m,n)B^*(m,n) \\
&= \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} A(m,n)[\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_B^*(w_1,w_2)e^{-j(mw_1+nw_2)}dw_1dw_2] \\
&= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}[\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} A(m,n)e^{-j(mw_1+nw_2)}]F_B^*(w_1,w_2)dw_1dw_2 \\
&= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_A(w_1,w_2)F_B^*(w_1,w_2)dw_1dw_2 \qquad (12)
\end{aligned}
$$

- Hence, energy conservation:

$$
\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} |A(m,n)|^2 = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |F_A(w_1,w_2)|^2 dw_1dw_2 \qquad (13)
$$

Convolution

- Let $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$.

$$\begin{aligned}
C(m,n) &= \sum_{k=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} A(k,l)B(m-k,n-l) \\
F_C(w_1,w_2) &= \sum_{k=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} A(k,l) \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} B(m-k,n-l)e^{-j(mw_1+nw_2)} \\
&= \sum_{k=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} A(k,l)F_B(w_1,w_2)e^{-j(kw_1+lw_2)} \\
&= F_B(w_1,w_2) \sum_{k=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} A(k,l)e^{-j(kw_1+lw_2)} \\
&= F_A(w_1,w_2)F_B(w_1,w_2)
\end{aligned}$$

where we used the shifting property in the second step of the calculation. Thus we have the important result:

$$\mathbf{A} \otimes \mathbf{B} \overset{\mathcal{F}}{\leftrightarrow} F_A(w_1,w_2)F_B(w_1,w_2) \tag{14}$$

Multiplication

- A dual property to convolution property can be derived for multiplication. Let $C(m,n) = A(m,n)B(m,n)$.

$$\mathcal{F}(C(m,n)) = \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} A(m,n)B(m,n)e^{-j(mw_1+nw_2)}$$

$$= \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} A(m,n)[\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_B(w_1',w_2')e^{j(mw_1'+nw_2')}dw_1'dw_2']e^{-j(mw_1+nw_2)}$$

$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_B(w_1',w_2')[\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} A(m,n)e^{-j(m(w_1-w_1')+n(w_2-w2'))}]dw_1'dw_2'$$

$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_B(w_1',w_2')F_A(w_1-w_1',w_2-w_2')dw_1'dw_2'$$

Thus:

$$A(m,n)B(m,n) \stackrel{\mathcal{F}}{\leftrightarrow} \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_B(w_1',w_2')F_A(w_1-w_1',w_2-w_2')dw_1'dw_2' \tag{15}$$

Delta Functions

- The Fourier transform of a Kronecker delta function:

$$\begin{aligned} F_\delta(w_1, w_2) &= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \delta(m,n) e^{-j(mw_1+nw_2)} \\ &= 1 \end{aligned} \quad (16)$$

- The Fourier transform of $A(m,n) = 1$ can be found via the Dirac delta function:

$$\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \delta(w_1, w_2) e^{j(mw_1+nw_2)} dw_1 dw_2 = \frac{1}{4\pi^2}$$

$$\Rightarrow A(m,n) = 1 \stackrel{\mathcal{F}}{\leftrightarrow} 4\pi^2 \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} \delta(w_1 - k2\pi, w_2 - l2\pi) \quad (17)$$

where $\delta(w_1, w_2)$ is the Dirac delta function and we used the fact that the Fourier transform has to be periodic with 2π.

- Note that $\delta(w_1, w_2) = 0$ for $w_1, w_2 \neq 0$ and

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(w_1, w_2) dw_1 dw_2 = 1 \quad (18)$$

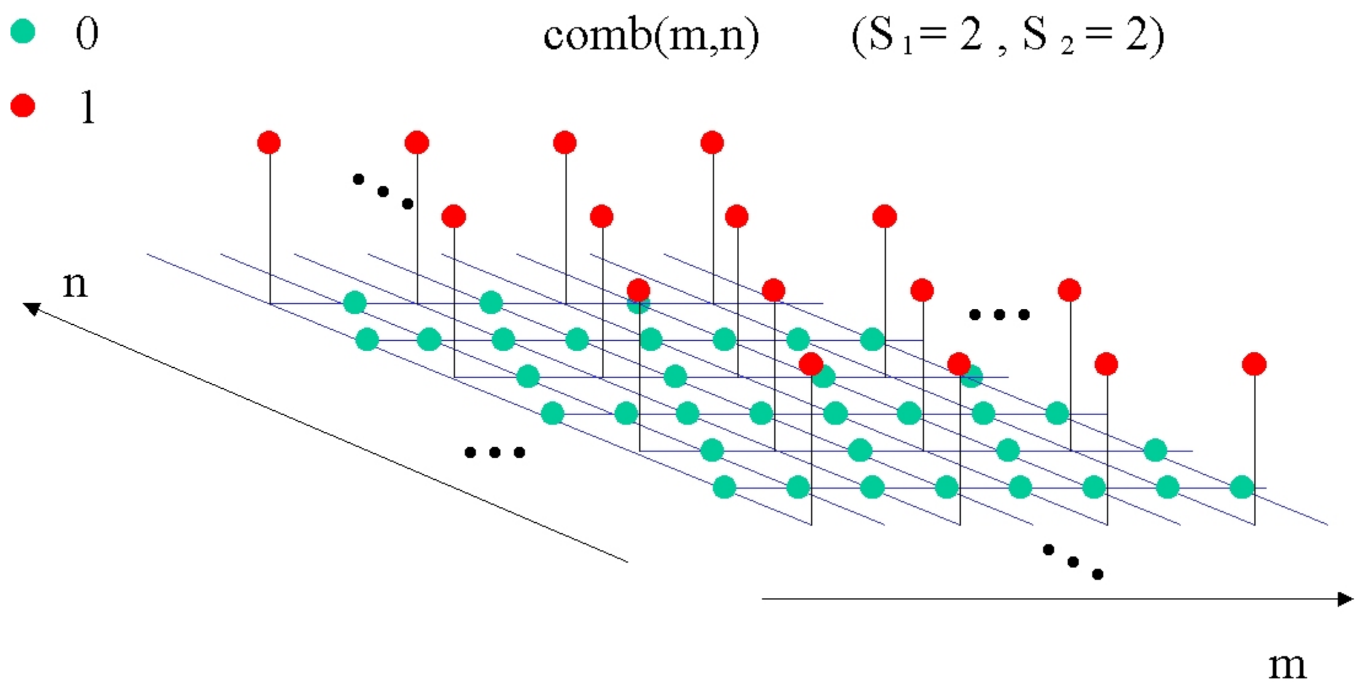
comb(m,n)

- Consider the Kronecker comb function $comb(m, n)$:

$$\sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} \delta(m - kS_1, n - lS_2) \tag{19}$$

where $S_1 > 0, \ S_2 > 0$ are integers.

- $comb(m, n)$ is very useful when discussing sampling.



comb(m,n) - contd.

The Fourier transform of a comb function can be computed as:

$$
\begin{aligned}
\mathcal{F}(comb(m,n)) &= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} comb(m,n) e^{-j(mw_1+nw_2)} \\
&= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} [\sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} \delta(m-kS_1, n-lS_2)] e^{-j(mw_1+nw_2)} \\
&= \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} [\sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \delta(m-kS_1, n-lS_2) e^{-j(mw_1+nw_2)}] \\
&= \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} 1 \; e^{-j(kS_1w_1+lS_2w_2)} \\
&= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} 1 \; e^{-j(mS_1w_1+nS_2w_2)}
\end{aligned}
\qquad (20)
$$

comb(m,n) - contd.

Note that Equation 20 is simply the Fourier transform of $A(m,n) = 1$ and hence:

$$\begin{aligned} \mathcal{F}(comb(m,n)) &= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} 1 \; e^{-j(mS_1w_1+nS_2w_2)} \\ &= 4\pi^2 \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} \delta(S_1w_1 - k2\pi, S_2w_2 - l2\pi) \\ &= \frac{4\pi^2}{S_1S_2} \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} \delta(w_1 - \frac{k2\pi}{S_1}, w_2 - \frac{l2\pi}{S_2}) \end{aligned} \quad (21)$$

where the last line follows since for any "regular" function $G(w_1, w_2)$:

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(S_1w_1 - k2\pi, S_2w_2 - l2\pi) G(w_1,w_2) dw_1 dw_2 &= \frac{1}{S_1S_2} G(k2\pi/S_1, l2\pi/S_2) \\ = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{S_1S_2} \delta(w_1 - \frac{k2\pi}{S_1}, w_2 - \frac{l2\pi}{S_2}) G(w_1,w_2) dw_1 dw_2 \end{aligned}$$

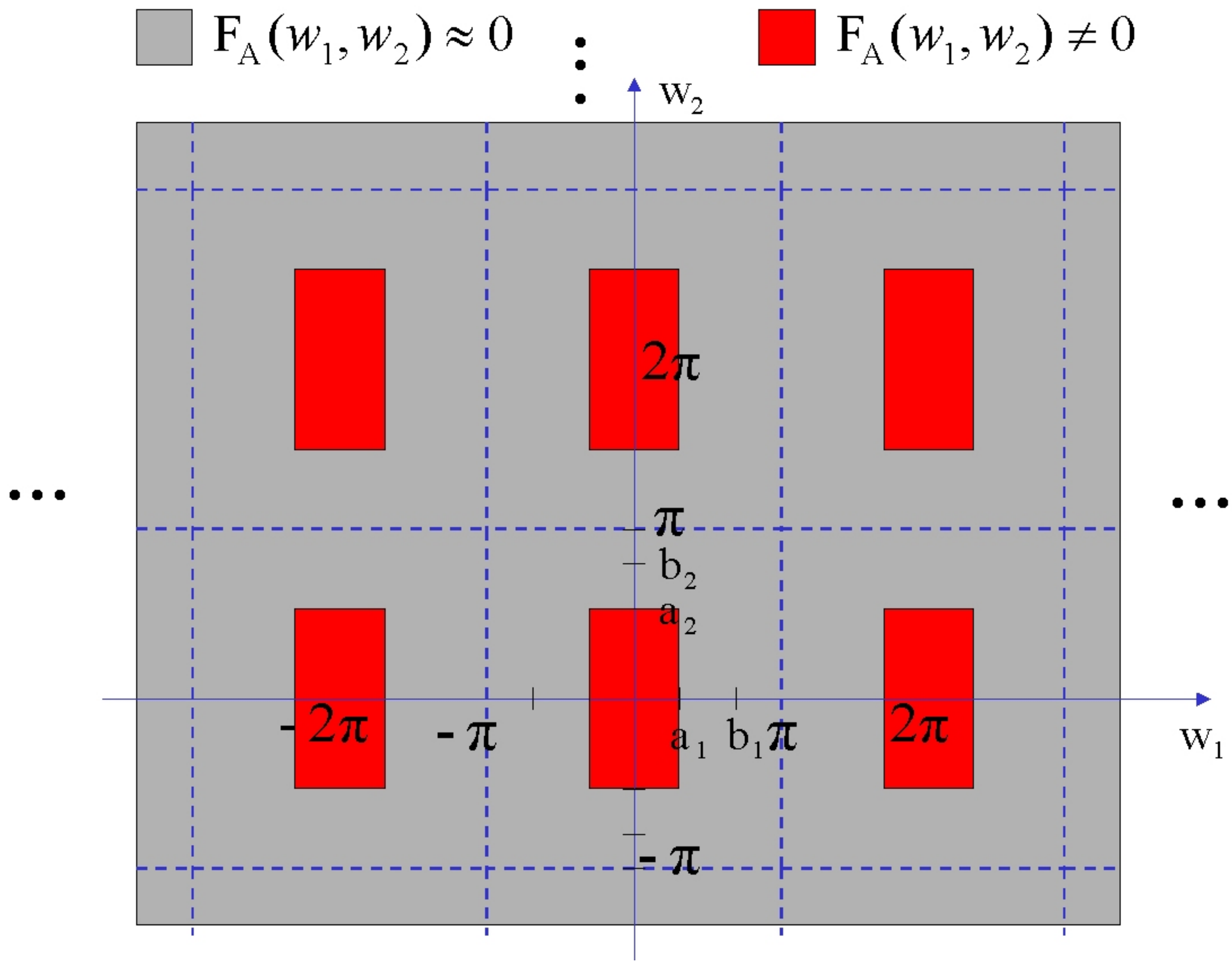
and Dirac delta functions are defined by integrals.

Fourier Transform Types

Let $0 < a_1 < b_1 < \pi$ and $0 < a_2 < b_2 < \pi$.

- We will say that a Fourier transform $F_A(w_1, w_2)$ is low pass if $|F_A(w_1, w_2)| \sim 0$ when $a_1 < |w_1| < \pi$ and $a_2 < |w_2| < \pi$.
- We will say that a Fourier transform $F_A(w_1, w_2)$ is high pass if $|F_A(w_1, w_2)| \sim 0$ when $0 < |w_1| < a_1$ and $0 < |w_2| < a_2$.
- Finally, we will say that a Fourier transform $F_A(w_1, w_2)$ is band pass if $|F_A(w_1, w_2)| \sim 0$ when $0 < |w_1| < a_1,\ b_1 < |w_1| < \pi$ and $0 < |w_2| < a_2,\ b_2 < |w_2| < \pi$.

Example - Low pass



Sampling and Aliasing

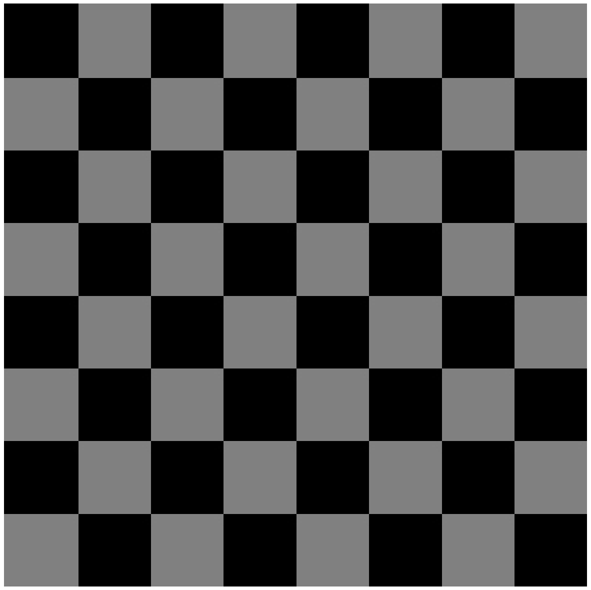
- Given a 2-D sequence $\mathbf{A}$ we would like to obtain a sequence $\mathbf{C}$ by sub-sampling $\mathbf{A}$:

$$C(m,n) = A(S_1 m, S_2 n) \tag{22}$$

  where $S_1, S_2 > 0$ are integers.

- We would like $\mathbf{C}$ to have close resemblance to $\mathbf{A}$.
- For example given a 512×512 image we would like to obtain a 256×256 image by picking every other pixel in the original image.
- Things may go very wrong in sampling with unexpected effects.

Example



Original image (8x8)

Two possible 4x4 sub-sampled images

Fourier Transform of Sampled Sequence

$$
\begin{aligned}
B(m,n) &= A(m,n)comb(m,n) \\
C(m,n) &= B(S_1 m, S_2 n)
\end{aligned}
$$

- First obtain the Fourier transform of B using the multiplication property:

$$
F_B(w_1,w_2) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_A(w_1',w_2')F_{comb}(w_1-w_1',w_2-w_2')dw_1'dw_2'
$$

$$
= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_A(w_1',w_2')[\frac{4\pi^2}{S_1S_2}\sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty}\delta(w_1-w_1'-\frac{k2\pi}{S_1},w_2-w_2'-\frac{l2\pi}{S_2})]dw_1'dw_2'
$$

$$
= \sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty}\frac{1}{S_1S_2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F_A(w_1',w_2')\delta(w_1-w_1'-\frac{k2\pi}{S_1},w_2-w_2'-\frac{l2\pi}{S_2})dw_1'dw_2'
$$

For $w_1, w_2 \in [-\pi,\pi)$, let $K(w_1) = \{k|w_1 - \frac{k2\pi}{S_1} \in [-\pi,\pi)\}$ and $L(w_2) = \{l|w_2 - \frac{l2\pi}{S_2} \in [-\pi,\pi)\}$. Then:

$$
F_B(w_1,w_2) = \frac{1}{S_1S_2}\sum_{k\in K(w_1)}\sum_{l\in L(w_2)} F_A(w_1-\frac{k2\pi}{S_1},w_2-\frac{l2\pi}{S_2}) \tag{23}
$$

Example

Suppose $S_1 = S_2 = 2$, i.e., we are sub-sampling by 2. Then for $w_1, w_2 \in (-\pi, \pi)$,

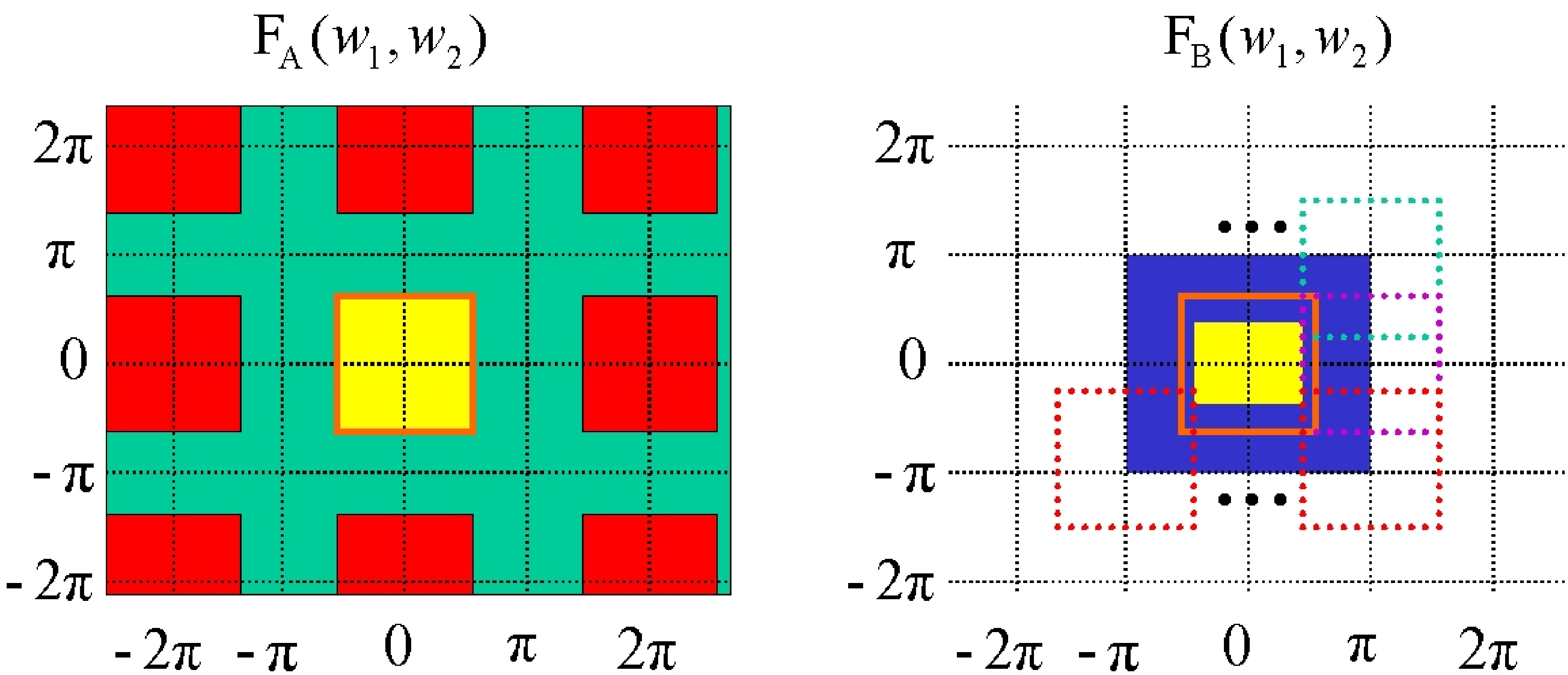
$$\begin{aligned} K(w_1) &= \{k | w_1 - k\pi \in (-\pi, \pi)\} = \{-1, 0, 1\} \\ L(w_2) &= \{l | w_2 - l\pi \in (-\pi, \pi)\} = \{-1, 0, 1\} \end{aligned}$$

and we have:

$$F_B(w_1, w_2) = \sum_{k=-1}^{1} \sum_{l=-1}^{1} F_A(w_1 - k\pi, w_2 - l\pi) \tag{24}$$

- If during this process there is overlapping, i.e., say $(F_B(w_1, w_2) - F_A(w_1, w_2))F_A(w_1, w_2) \neq 0$ then we will say that there is aliasing in the sub-sampling operation.

Example - contd.



■ Aliased Frequencies

Aliased frequencies shown inside [- π, π) x [- π, π) only

F-T of Sampled Sequence - contd.

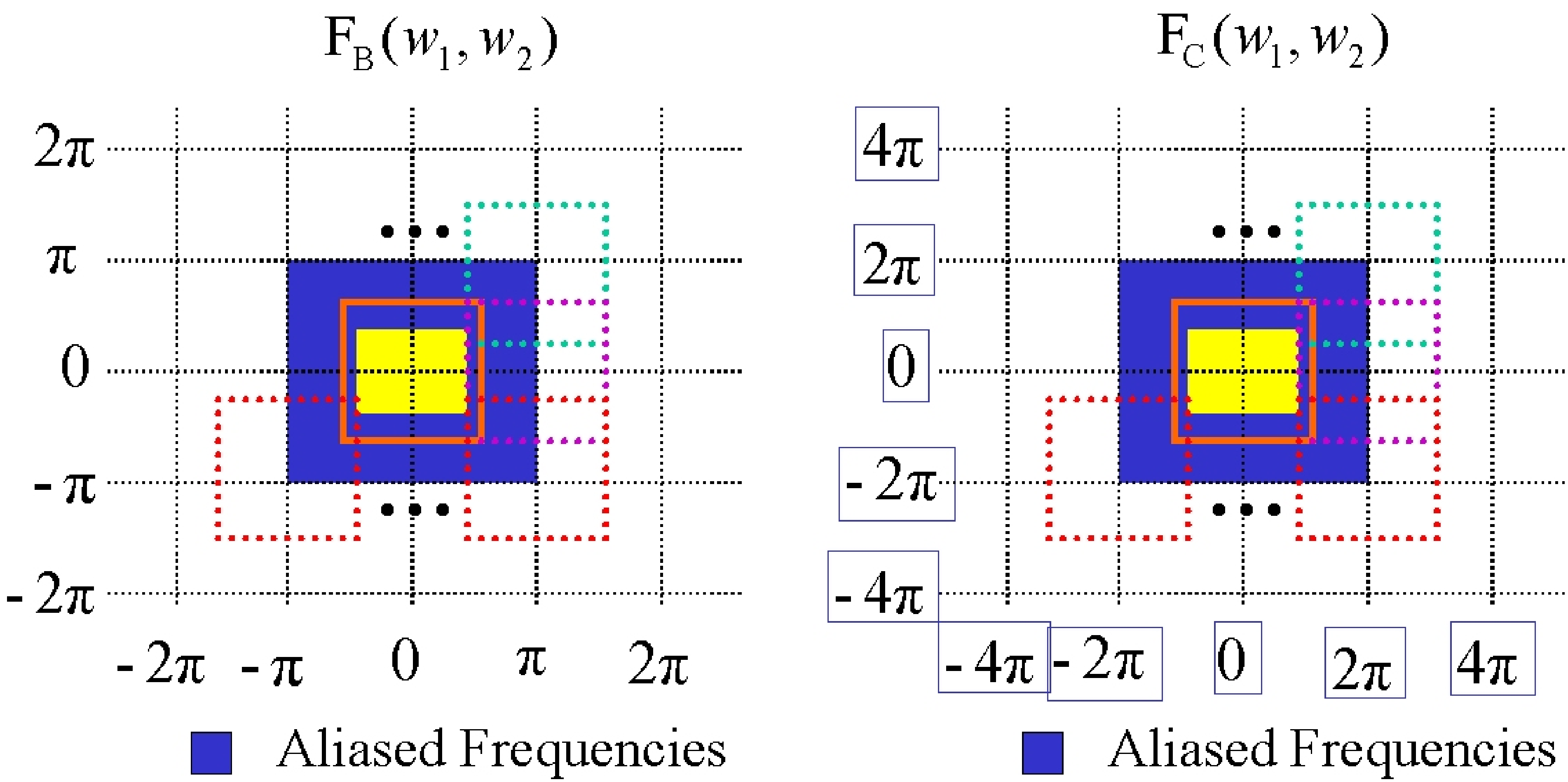
Going back to the transform we were calculating:

- We can now obtain $F_C(w_1, w_2)$

$$
\begin{aligned}
F_C(w_1, w_2) &= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} B(S_1 m, S_2 n) e^{-j(mw_1 + nw_2)} \\
&= \sum_{m=\ldots,-S_1,0,S_1,\ldots} \sum_{n=\ldots,-S_2,0,S_2,\ldots} B(m, n) e^{-j(mw_1/S_1 + nw_2/S_2)} \\
&= F_B(w_1/S_1, w_2/S_2) \\
&= \frac{1}{S_1 S_2} \sum_{k \in K(w_1)} \sum_{l \in L(w_2)} F_A\left(\frac{w_1}{S_1} - \frac{k2\pi}{S_1}, \frac{w_2}{S_2} - \frac{l2\pi}{S_2}\right) \qquad (25)
\end{aligned}
$$

Example - contd.

$$S_1 = S_2 = 2$$



Aliasing

- It is clear that unless we are careful, the sampled signal can be very different from the original.
- For no aliasing to occur after sampling $F_A(w_1, w_2)$ must be:

$$F_A(w_1, w_2) = 0, \ \frac{\pi}{S_1} < |w_1| < \pi, \ \frac{\pi}{S_2} < |w_2| < \pi \tag{26}$$

  so that there is no overlap.
- But what if there is?
  - Then we have to low-pass filter $\mathbf{A}$ to make sure things become conformant to the above.
  - Such a low-pass filter is called an antialiasing filter.

Summary

- In this lecture we learnt the equivalence between convolution and linear filtering.
- We reviewed two dimensional Fourier transforms of 2-d sequences.
- We discussed various properties of Fourier transforms and in particular we saw that the Fourier transform "converts" convolution to multiplication.
- Using the Fourier transform properties of Kronecker and Dirac delta functions we learnt about sampling and aliasing.

Homework VI

1. Show the modulation property of the Fourier transform.

2. Show that if a two dimensional sequence is separable then so is its Fourier transform, i.e., if $A(m,n) = A_1(m)A_2(n)$ then $F_A(w_1,w_2) = F_{A_1}(w_1)F_{A_2}(w_2)$ where $F_{A_1}(w_1), F_{A_2}(w_2)$ are *one* dimensional Fourier transforms such as $F_a(w) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} a(n)e^{-jwn}$.

3. Obtain the Fourier transform of the 2-D sequence $A(m,n)$ given by:

	$n=0$	1	2
$m=0$	1	2	-1
1	2	4	-2
2	-1	-2	1

Simplify your answer as much as possible.

4. Calculate the Fourier transform of the limited extent sequence $A(m,n) = 1,\ 0 \le m < 8, 0 \le n < 8$ and $A(m,n) = 0$ otherwise.

5. Calculate the Fourier transform of the limited extent sequence $A(m,n) = (-1)^{m+n},\ 0 \le m < 8, 0 \le n < 8$ and $A(m,n) = 0$ otherwise.

6. Based on the two items above, find the Fourier transform of the checkerboard image($D(m,n)$). (Hint: Assume gray=2, black=0). For sampling with $S_1 = S_2 = 2$ show that there is aliasing. Calculate the Fourier transform of the sub-sampled sequence $C(m,n) = D(S_1 m, S_2 n)$. (Do this by using the $F_D(w_1,w_2)$. Take the inverse transform and verify with direct sub-sampling in the "sequence domain".

7. Find the periodicity of $F_B(w_1, w_2)$ and $F_C(w_1, w_2)$ as obtained in sampling and aliasing slides. Draw a figure showing *both* the aliasing and periodicity involved in $F_B(w_1, w_2)$, $F_C(w_1, w_2)$. Take $F_A(w_1, w_2)$ of the original sequence and S_1, S_2 anything you like as long as there is antialiasing. Do a better job than I did.

References

[1] A. K. Jain, *Fundamentals of Digital Image Processing.* Englewood Cliffs, NJ: Prentice Hall, 1989.